An extension of Markov's Theorem

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Abstract

We give a general sufficient condition for the uniform convergence of sequences of type II Hermite-Padé approximants associated with Nikishin systems of functions.

1. Introduction

Let $\Delta \subset \mathbb{R}$ be a compact interval and $\mathcal{M}(\Delta)$ the set of finite Borel measures with constant sign whose support $S(\mu)$ is a subset of Δ such that Δ is the smallest interval which contains $S(\mu)$; we write $\mathrm{Co}(S(\mu)) = \Delta$. Given $\mu \in \mathcal{M}(\Delta)$, the associated Markov function is defined by

$$\widehat{\mu}(z) = \int \frac{d\mu(x)}{z - x} \in \mathcal{H}(\overline{\mathbb{C}} \setminus S(\mu))$$

which is holomorphic in $\overline{\mathbb{C}} \setminus S(\mu)$.

Fix a measure $\sigma \in \mathcal{M}(\Delta)$ and a system of m weights $\mathbf{r} = (\rho_1, \ldots, \rho_m)$ with respect to σ; that is, each $\rho_k \in L_1(\sigma)$ and has constant sign. Consider the system of measures $\mathbf{s} = (s_1, \ldots, s_m)$, where $ds_j = \rho_j d\sigma$, and the corresponding system of Markov functions $\widehat{\mathbf{s}} = (\widehat{s}_1, \ldots, \widehat{s}_m)$. Take a multi-index

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$\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$, where $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. There exist polynomials $Q_{\mathbf{n}}$ and $P_{\mathbf{n},j}$, $j = 1, \ldots, m$, such that

$$
\begin{aligned}
&i) \quad \deg Q_{\mathbf{n}} \leq |\mathbf{n}| = n_1 + \cdots + n_m, \qquad Q_{\mathbf{n}} \not\equiv 0, \\
&ii) \quad (Q_{\mathbf{n}} \widehat{s}_j - P_{\mathbf{n},j})(z) = \mathcal{O}\left(1/z^{n_j+1}\right), \qquad z \to \infty, \qquad j = 1, \ldots, m.
\end{aligned} \tag{1.1}
$$

In the sequel we assume that $Q_{\mathbf{n}}$ is monic.

For each $j = 1, \ldots, m$, $Q_{\mathbf{n}}$ annihilates the terms corresponding to the powers between -1 and $-n_j$ of the Laurent expansion of $Q_{\mathbf{n}} \widehat{s}_j$ whereas $P_{\mathbf{n},j}$ represents the polynomial part of $Q_{\mathbf{n}} \widehat{s}_j$. Hence, $Q_{\mathbf{n}}$ determines univocally $P_{\mathbf{n},j}$ and, consequently, the rational fraction $P_{\mathbf{n},j}/Q_{\mathbf{n}}$.

The vector rational fractions $\mathbf{R}_{\mathbf{n}} = (P_{\mathbf{n},1}/Q_{\mathbf{n}}, \ldots, P_{\mathbf{n},m}/Q_{\mathbf{n}})$ is called type II Hermite-Padé approximant corresponding to the system $\widehat{\mathbf{s}}$ and the multi-index $\mathbf{n}$.

When $m = 1$, $\mathbf{R}_{\mathbf{n}} = P_{\mathbf{n},1}/Q_{\mathbf{n}} = P_n/Q_n$, $\mathbf{n} = n$, is the nth diagonal Padé approximant of $\widehat{s}_1 = \widehat{s}$. It is well known (for example, see Chapter II in [15]), that in this case Q_n is the nth monic orthogonal polynomial with respect to the measure s. Usually, monic orthogonal polynomials are defined for positive measures, however, the definition is trivially extended to measures with constant sign. Q_n has n simple zeros in the interior of $\mathrm{Co}(S(s))$ (see [16, Lemma 1.1.3]).

In [13], A.A. Markov proved that given an arbitrary measure $s \in \mathcal{M}(\Delta)$ the sequence $\{P_n/Q_n\}_{n \in \mathbb{Z}_+}$ converges uniformly to $\widehat{s}$ on every compact subset contained in the domain $\overline{\mathbb{C}} \setminus \Delta$. We write

$$
\frac{P_n}{Q_n} \underset{n \to \infty}{\rightrightarrows} \widehat{s}, \qquad \text{on} \qquad \overline{\mathbb{C}} \setminus \Delta.
$$

In the present paper, we extend Markov's Theorem to the context of type II Hermite-Padé approximation.

The first drawback in extending Markov's Theorem to the context of Hermite-Padé approximation is that in the vector case, in general, $Q_{\mathbf{n}}$ is not uniquely determined by (1.1). However, in [10] it is shown that uniqueness takes place for the so called Nikishin systems of measures which we introduce below. In this case, $Q_{\mathbf{n}}$ also has $|\mathbf{n}|$ simple zeros in the interior of Δ.

Nikishin systems of measures were introduced by E.M. Nikishin in his famous article [14]. Take two compact intervals Δ_α and Δ_β of the real line such that $\Delta_\alpha \cap \Delta_\beta = \emptyset$ and two measures $\sigma_\alpha \in \mathcal{M}(\Delta_\alpha)$ and $\sigma_\beta \in \mathcal{M}(\Delta_\beta)$. We define a third measure $\langle \sigma_\alpha, \sigma_\beta \rangle$ whose differential expression is

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) = \int \frac{d\sigma_\beta(t)}{x-t} d\sigma_\alpha(x) = \widehat{\sigma}_\beta(x) d\sigma_\alpha(x).$$

Observe that $\langle \sigma_\alpha, \sigma_\beta \rangle \in \mathcal{M}(\Delta_\alpha)$.

Now, take m compact intervals $\Delta_1, \ldots, \Delta_m$ with the property that for each $j = 1, \ldots, m-1$, $\Delta_j \cap \Delta_{j+1} = \emptyset$. Let $(\sigma_1, \ldots, \sigma_m)$ be a system of measures such that $\sigma_j \in \mathcal{M}(\Delta_j)$, $j = 1, \ldots, m$. The system of measures $(s_1, \ldots, s_m)$ given by

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \quad s_3 = \langle \sigma_1, \langle \sigma_2, \sigma_3 \rangle \rangle = \langle \sigma_1, \sigma_2, \sigma_3 \rangle, \ldots, s_m = \langle \sigma_1, \ldots, \sigma_m \rangle,$$

is the so called Nikishin system of measures generated by $(\sigma_1, \ldots, \sigma_m)$. For short, we write $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ whereas $\widehat{\mathbf{s}} = (\widehat{s}_1, \ldots, \widehat{s}_m) = \widehat{\mathcal{N}}(\sigma_1, \ldots, \sigma_m)$ is the corresponding Nikishin system of functions. Nikishin systems have received a great deal of attention in the recent past and have found numerous applications, see for example [1], [2], [3], [4], [6], [7] [8], [11], [12] and [17].

In order to state our main result we need to review some concepts. Given two disjoint compact sets K_1 and K_2 of $\mathbb{R}$, $\operatorname{dist}(K_1, K_2)$ denotes the distance between K_1 and K_2 i.e. $\operatorname{dist}(K_1, K_2) = \min\{|x_1 - x_2| : (x_1, x_2) \in K_1 \times K_2\}$ whereas $\operatorname{diam}(K_1) = \max\{|x_1 - x_2| : x_1, x_2 \in K_1\}$ denotes the diameter of K_1.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $\{\mathbf{R_n} = (P_{\mathbf{n},1}/Q_{\mathbf{n}}, \ldots, P_{\mathbf{n},m}/Q_{\mathbf{n}})\}_{\mathbf{n}\in\mathbf{\Lambda}}$ be the sequence of type II Hermite-Padé approximants corresponding to a sequence of distinct multi-indices $\mathbf{\Lambda} \subset \mathbb{Z}_+^m$ and a system $(\widehat{s}_1, \ldots, \widehat{s}_m) = \widehat{\mathcal{N}}(\sigma_1, \ldots, \sigma_m)$. Assume* $\operatorname{diam}(\Delta_k) < \operatorname{dist}(\Delta_1, \Delta_2)$. *Then, for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$*

$$\limsup_{\mathbf{n}\in\Lambda} \left|\left| \widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} \right|\right|_K^{1/(|\mathbf{n}|+n_j)} \leq ||\phi_\infty||_K < 1, \qquad j = 1, \ldots, m,$$

where $||\cdot||_K$ denotes the sup-norm on K and ϕ_∞ denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_1$ onto the open unit disk such that $\phi_\infty(\infty) = 0$ and $\phi'_\infty(\infty) > 0$.

Notice that the sequence of multi-indices may be completely arbitrary. In Markov's Theorem, there is no assumption on the measure. This is also true in our case whenever $\mathrm{diam}(\Delta_k) < \mathrm{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$. We have imposed no restrictions on the measures $\sigma_3, \ldots, \sigma_m$ at all. Another extension of Markov's Theorem was given in [10, Corollary 1.1] without any assumption on the measures, but the indices are required to satisfy $n_j \geq |\mathbf{n}|/m - c|\mathbf{n}|^\kappa$, $j = 1, \ldots, m$, for $c > 0$ and $\kappa < 1$. We believe that a complete analogue of Markov's Theorem should hold.

The following result extends [9, Corollary 2] to a larger class of multi-indices.

Theorem 1.2. *Let $\mathbf{\Lambda} \subset \mathbb{Z}_+^m$ be a sequence of multi-indices such that either there exists $k \in \{2, \ldots, m\}$ such that for every $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbf{\Lambda}$, $n_k = \max\{n_1 + 1, n_2, \ldots, n_m\}$, or $n_1 = \max\{n_1, n_2 - 1, \ldots, n_m - 1\}$ (in which case we take $k = 1$). Then, for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$,*

$$\limsup_{\mathbf{n} \in \Lambda} \left|\left| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right|\right|_K^{1/2|\mathbf{n}|} \leq \kappa(K) < 1, \tag{1.2}$$

where

$$\kappa(K) = \sup\{||\phi_t||_K : t \in \Delta_2 \cup \{\infty\}\}$$

and ϕ_t denotes the conformal representation of $\overline{\mathbb{C}} \setminus \Delta_1$ onto the open unit disk such that $\phi_t(t) = 0$ and $\phi_t'(t) > 0$.

In the first three sections we give some preliminary results which are necessary for the proof of the Theorems above. Section 2 includes some properties of multiple orthogonal polynomials corresponding to Nikishin systems of measures. In Section 3 we study properties of Fourier series of functions expanded in terms of orthogonal polynomials with respect to varying measures. Theorem 1.2 is proved in Section 4 as a first step to the proof of Theorem 1.1 which is completed in Section 5.

2. Multiple orthogonality in Nikishin systems

Let $\mathbf{s} = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ and $\mathbf{n} = (n_1, \ldots, n_m)$ be given. It is well known and easy to verify that the conditions (1.1) imply

$$0 = \int x^\nu Q_{\mathbf{n}}(x) ds_j(x), \quad \nu = 0 \ldots, n_j - 1, \quad j = 1, \ldots, m. \tag{2.1}$$

For each $j = 1, \ldots, m$, let h be an arbitrary polynomial such that $\deg h \le n_j$. Then

$$0 = \int \frac{h(z) - h(x)}{z - x} Q_{\mathbf{n}}(x) ds_j(x) \tag{2.2}$$

hence

$$\int \frac{Q_{\mathbf{n}}(x)}{z - x} ds_j(x) = \frac{1}{h(z)} \int \frac{h(x) Q_{\mathbf{n}}(x)}{z - x} ds_j(x) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \quad \text{as} \quad z \to \infty.$$

Define

$$P(z) = \int \frac{Q_{\mathbf{n}}(z) - Q_{\mathbf{n}}(x)}{z - x} ds_j(x).$$

Thus

$$\left(Q_{\mathbf{n}} \widehat{s}_j - P\right)(z) = \int \frac{Q_{\mathbf{n}}(x)}{z - x} ds_j(x) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \quad \text{as} \qquad z \to \infty.$$

From (1.1) we see that

$$P(z) - P_{\mathbf{n},j}(z) = \mathcal{O}\left(\frac{1}{z^{n_j+1}}\right) \in \mathcal{H}\left(\overline{\mathbb{C}}\right) \quad z \to \infty.$$

Consequently,

$$P_{\mathbf{n},j}(z) \equiv \int \frac{Q_{\mathbf{n}}(z) - Q_{\mathbf{n}}(x)}{z - x} ds_j(x), \quad \left(Q_{\mathbf{n}} \widehat{s}_j - P_{\mathbf{n},j}\right)(z) = \int \frac{Q_{\mathbf{n}}(x)}{z - x} ds_j(x). \tag{2.3}$$

From [10] we know that the conditions (2.1) imply that $Q_{\mathbf{n}}$ has $|\mathbf{n}|$ simple zeros which lie in the interior of Δ_1. Let $x_{\mathbf{n},1} < \ldots < x_{\mathbf{n},|\mathbf{n}|}$ be the zeros of $Q_{\mathbf{n}}$. Decomposing into simple fractions, we get

$$\frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} = \sum_{i=1}^{|\mathbf{n}|} \frac{\lambda_{i,j,\mathbf{n}}}{z - x_{\mathbf{n},i}}, \qquad j = 1, \ldots, m. \tag{2.4}$$

The coefficients $\lambda_{i,j,\mathbf{n}}$, $i = 1, \ldots, |\mathbf{n}|$ and $j = 1, \ldots, m$, were called Nikishin-Christoffel coefficients in [9, Definition 2]. Taking into account the equality in (2.3), we have that

$$\lambda_{i,j,\mathbf{n}} = \lim_{z \to x_{\mathbf{n},i}} (z - x_{\mathbf{n},i}) \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} = \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})}. \tag{2.5}$$

For each $j = 1, \dots, m$,

$$\left|\sum_{i=1}^{|\mathbf{n}|} \lambda_{i,j,\mathbf{n}}\right| = \left|\sum_{i=1}^{|\mathbf{n}|} \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})}\right| = \tag{2.6}$$

$$\left|\int \sum_{i=1}^{|\mathbf{n}|} \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} ds_j(x)\right| = \left|\int ds_j(x)\right| = ||s_j|| < +\infty,$$

where $||s||$ represents the total variation of the measure s. In this chain of equalities we have used that $\mathcal{P}(x) = \sum_{i=1}^{|\mathbf{n}|} Q_{\mathbf{n}}(x) / \left(Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x_{\mathbf{n},i} - x)\right)$ is the polynomial of degree $\leq |\mathbf{n}| - 1$ which interpolates the constant function 1 at the zeros of $Q_{\mathbf{n}}$. Thus $\mathcal{P} \equiv 1$.

From [10, Lemma 3.2] one can state the following result. (We wish to point out that the measure denoted here with τ are products of those in [10].)

Lemma 2.1. *Let* $(\widehat{s}_{2,2}, \dots, \widehat{s}_{2,m}) = \widehat{\mathcal{N}}(\sigma_2, \dots, \sigma_m)$*, there is a system of* $m-1$ *measures* $(\tau^k_{2,1}, \dots, \tau^k_{2,k-1}, \tau^k_{2,k+1}, \dots, \tau^k_{2,m})$ *where* $Co(S(\tau^k_{2,j})) \subset \Delta_2$, $j = 1, \dots, k-1, k+1, \dots, m$*, such that*

$$\frac{1}{\widehat{s}_{2,k}(z)} = \ell_{2,k}(z) + \widehat{\tau}^k_{2,1}(z), \tag{2.7}$$

where $\ell_{2,k}$ *denotes a polynomial with degree one, and*

$$\frac{\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)} - \frac{|s_{2,j}|}{|s_{2,k}|} = \widehat{\tau}^k_{2,j}(z), \quad j = 2, \dots, k-1, k+1, \dots, m. \tag{2.8}$$

Theorem 1.4 in [10] refers to so called mixed type multiple orthogonal polynomials of two Nikishin systems. When reduced to type II multiple orthogonal polynomials of a Nikishin system it may be restated in the following form.

Lemma 2.2. *Let* $(s_1, \dots, s_m) = \mathcal{N}(\sigma_1, \dots, \sigma_m)$ *and* $\mathbf{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m_+$ *be given. Set* $k = 1$ *if* $n_1 + 1 = M = \max\{n_1 + 1, n_2 \dots n_m\}$*, otherwise* k *is equal to the subscript of the first component of* $\mathbf{n}$ *such that* $M = n_k$*. Then, there exists a permutation* λ *of* $\{1, \dots, m\}$ *which reorders the components of* $\mathbf{n}$ *such that* $n_{\lambda(1)} + \delta_{\lambda(1),1} \geq n_{\lambda(2)} \geq \cdots \geq n_{\lambda(m)}$ *with* $n_k = n_{\lambda(1)}$ *and* $\delta_{\lambda(1),1}$ *denoting the known Kronecker delta function, and an associated*

Nikishin system $\widetilde{\mathbf{s}} = (r_1, \ldots, r_m) = \mathcal{N}(\rho_1, \ldots, \rho_m)$, *where* $s_k = r_1 = \rho_1$ *and* $\mathrm{Co}(S(\rho_j)) \subset \Delta_j$, $j = 1, \ldots, m$, *such that if* $\widetilde{\mathbf{n}} = (n_{\lambda(1)}, \ldots, n_{\lambda(m)})$, *the pairs* $(\mathbf{s}, \mathbf{n})$ *and* $(\widetilde{\mathbf{s}}, \widetilde{\mathbf{n}})$ *have the same type II multiple orthogonal polynomial. That is,* $Q_{\mathbf{n}}$ *satisfies (2.1) and*

$$0 = \int x^\nu Q_{\mathbf{n}}(x) \widehat{r}_{2,j}(x) ds_k(x), \quad \nu = 0 \ldots, n_{\lambda(j)} - 1, \quad j = 1, \ldots, m, \tag{2.9}$$

where $r_{2,j} = \langle \rho_2, \ldots, \rho_j \rangle, j = 2, \ldots, m$, *and* $\widehat{r}_{2,1} \equiv 1$.

Type II multiple orthogonal polynomials of Nikishin systems with respect to decreasing multi-indices satisfy other orthogonality relations. In particular, from Propositions 2 and 3 in [11] (see also relations (5)-(7) in [2]), we have

Lemma 2.3. *Let* $\mathbf{s} = (s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $\mathbf{n} = (n_1, \ldots, n_m)$ *be given. Let* $k \in \{1, \ldots, m\}$ *be as in Lemma 2.2. Then, there exist two monic polynomial* $Q_{\mathbf{n},2}$, $\deg Q_{\mathbf{n},2} = |\mathbf{n}| - n_k$, *and* $Q_{\mathbf{n},3} = |\mathbf{n}| - n_k - n_{\lambda(2)}$, *whose zeros are simple and lie in the interior of* Δ_2 *and* Δ_3, *respectively, such that:*

$$\left(\frac{Q_{\mathbf{n}} \widehat{s}_k - P_{\mathbf{n},k}}{Q_{\mathbf{n},2}}\right)(z) = \mathcal{O}\left(\frac{1}{z^{|\mathbf{n}|+1}}\right) \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus S(\sigma_1)\right), \tag{2.10}$$

$$0 = \int x^\nu Q_{\mathbf{n}}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)}, \qquad \nu = 0, \ldots, |\mathbf{n}| - 1 \tag{2.11}$$

and

$$0 = \int t^\nu Q_{\mathbf{n},2}(t) \int \frac{Q^2_{\mathbf{n}}(x)}{t - x} \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t) Q_{\mathbf{n},3}(t)}, \qquad \nu = 0, \ldots, |\mathbf{n}| - n_k - 1. \tag{2.12}$$

(Here, ρ_2 *is the measure coming from Lemma 2.2.)*

Formulas (2.11) and (2.12) state that $Q_{\mathbf{n}}$ and $Q_{\mathbf{n},2}$ are the $|\mathbf{n}|$th and $(|\mathbf{n}| - n_k)$th monic orthogonal polynomials with respect to the varying measures

$$\frac{ds_k}{Q_{\mathbf{n},2}} \quad \text{and} \quad \int \frac{Q^2_{\mathbf{n}}(x)}{t - x} \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t) Q_{\mathbf{n},3}(t)}, \quad \text{respectively.} \tag{2.13}$$

There are other full orthogonality relations with respect to varying measures satisfied deeper in the system, but we will not need them.

3. Varying measures and associated Fourier series

Let sign : $\mathbb{R} \setminus \{0\} \to \{-1, 1\}$ denote the sign function. Analogously, $\text{sign}(\mu)$ will denote the sign of a given measure $\mu \in \mathcal{M}(\Delta)$. Notice that $\text{sign}(\mu) \cdot \mu$ is a positive measure. Given a measurable function $f : \Delta \to \mathbb{R}$,

$$||f||_{2,\mu} = \sqrt{\text{sign}(\mu) \int f^2(x) d\mu(x)},$$

denotes the L_2 norm with respect to μ. If $||f||_{2,\mu} < +\infty$ we write $f \in \mathrm{L}_2(\mu)$.

Let $\{q_{\mu,n}\}_{n \in \mathbb{Z}_+}$ be the family of monic orthogonal polynomials with respect to μ. For each $n \in \mathbb{Z}_+$ let $p_{\mu,n}(z) \equiv q_{\mu,n} / \, ||q_{\mu,n}||_{2,\mu}$ denote the nth orthonormal polynomial with respect to the measure μ. That is

$$\int p_{\mu,n}(x) p_{\mu,k}(x) d\mu(x) = \delta_{n,k} = \begin{cases} 1 & \text{if} \quad n = k \\ & \\ 0 & \text{if} \quad n \neq k \end{cases}, \qquad (n,k) \in \mathbb{Z}_+^2.$$

Fix $n \in \mathbb{Z}_+$, for each polynomial h of degree $\leq n$ we have the identity

$$0 = \int \frac{h(z) - h(x)}{z - x} p_{\mu,n}(x) d\mu(x),$$

thus

$$\int \frac{p_{\mu,n}(x) d\mu(x)}{z - x} = \frac{1}{p_{\mu,n}(z)} \int \frac{p^2_{\mu,n}(x) d\mu(x)}{z - x}. \tag{3.1}$$

From (2.11) we see that $q_{\mu,|\mathbf{n}|} \equiv Q_{\mathbf{n}}$ when $d\mu = ds_k / Q_{\mathbf{n},2}$, and $p_{\mu,|\mathbf{n}|} \equiv Q_{\mathbf{n}} / \, ||Q_{\mathbf{n}}||_{2,\mu}$.

Lemma 3.1. *Let $\{d\mu_n\}_{n \in \mathbb{Z}_+} \subset \mathcal{M}(\Delta)$ be given. Then for each $t \in \mathbb{C} \setminus \Delta$ we have that*

$$\left| \frac{q_{\mu_n,n}(x)}{q_{\mu_n,n}(t)} \right|^{1/n} \leq \frac{\text{diam}(\Delta)}{\text{dist}(t, \Delta)}, \qquad n \in \mathbb{Z}_+, \tag{3.2}$$

uniformly in $\{x \in \Delta\}$.

Proof. Fix $n \in \mathbb{Z}_+$. Since $q_{\mu_n,n}$ has its n zeros in the interior of Δ then

$$\left| \frac{q_{\mu_n,n}(x)}{q_{\mu_n,n}(z)} \right| \leq \left(\frac{\text{diam}(\Delta)}{\text{dist}(K, \Delta)} \right)^n.$$

This proves immediately (3.2). □

Fix two integers $n, \nu \in \mathbb{Z}_+$ and a function $f \in \mathrm{L}_2(\mu_\nu)$. The sum

$$S_{f,n,\mu_\nu}(z) = \sum_{i=0}^{n} \gamma_{i,\nu} p_{\mu_\nu,i}(z), \tag{3.3}$$

where

$$\gamma_{i,\nu} = \mathrm{sign}(\mu_\nu) \int f(x) p_{\mu_\nu,i}(x) d\mu_\nu(x), \qquad i = 0, \dots, n,$$

defines the nth partial sum of the Fourier series corresponding to f in terms of the orthonormal system $\{p_{\mu_n,i}\}_{i\in\mathbb{Z}_+}$.

Substituting in (3.3) the well known Christoffel-Darboux identity (Theorem 4.5 in [5] page 23) we obtain

$$S_{f,n,\mu_\nu}(z) = a_{\mu_\nu,n+1} \int \frac{p_{\mu_\nu,n+1}(z)p_{\mu_\nu,n}(x) - p_{\mu_\nu,n+1}(x)p_{\mu_\nu,n}(z)}{z-x} f(x) d\mu_\nu(x), \tag{3.4}$$

where

$$a_{\mu_\nu,n+1} = \int x p_{\mu_\nu,n+1}(x) p_{\mu_\nu,n}(x) d\mu_\nu(x).$$

Notice that $\mathrm{sign}(a_{\mu_n,n+1}) = \mathrm{sign}(\mu_n)$. For an arbitrary polynomial $\mathcal{P}$ of degree $\leq n$, $S_{\mathcal{P},n,\mu_n} \equiv \mathcal{P}$.

Proposition 3.1. *Let $\{\mu_n\}_{n\in\mathbb{Z}_+} \subset \mathcal{M}(\Delta)$ be given. Fix $t \in \mathbb{C} \setminus \Delta$ such that* $\mathrm{dist}(t,\Delta) > \mathrm{diam}(\Delta)$. *Then*

$$S_{1/(z-t),n,\mu_n} \rightrightarrows \frac{1}{z-t}, \quad \textit{for} \quad z \in \Delta. \tag{3.5}$$

Proof. Fix $N \in \mathbb{Z}_+$. We start by proving

$$S_{1/(z-t),n,\mu_N} \rightrightarrows \frac{1}{z-t}, \quad \text{for} \quad z \in \Delta. \tag{3.6}$$

For two nonnegative integers $n > n'$ we analyze the difference

$$\varepsilon_{N,n,n'} = \left|S_{1/(z-t),n',\mu_N} - S_{1/(z-t),n,\mu_N}\right| = \left|\sum_{i=n'+1}^{n} \gamma_{i,N} p_{\mu_N,i}(z)\right|, \tag{3.7}$$

where $\gamma_{i,N} = \int p_{\mu_N,i}(x)/(x-t) d\mu_N(x)$. So

$$\varepsilon_{N,n,n'} = \left|\sum_{i=n'+1}^{n} p_{\mu_N,i}(z) \int \frac{p_{\mu_N,i}(x)}{\rho_N(x)} \frac{d\mu_N(x)}{x-t}\right|.$$

Taking into account the identity given in (3.1) we have that

$$\varepsilon_{N,n,n'} = \left| \sum_{i=n'+1}^{n} \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \int \frac{p^2_{\mu_N,i}(x)d\mu_N(x)}{x-t} \right| \leq$$

$$\sum_{i=n'+1}^{n} \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right| \left| \int \frac{p^2_{\mu_N,i}(x)d\mu_N(x)}{x-t} \right| \leq \sum_{i=n'+1}^{n} \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right| \frac{\left|\int p^2_{\mu_n,i}(x)d\mu_N(x)\right|}{\mathrm{dist}(t,\Delta)}.$$

Hence we obtain that

$$\varepsilon_{N,n,n'} \leq \frac{1}{\mathrm{dist}(t,\Delta)} \sum_{i=n'+1}^{n} \left| \frac{p_{\mu_N,i}(z)}{p_{\mu_N,i}(t)} \right|.$$

Lemma 3.1 implies that there exists a nonnegative integer N' such that for every pair (n,n'), with $n \geq n' \geq N'$

$$\varepsilon_{N,n,n'} \leq \varepsilon_{N,n,n'} \leq \frac{1}{\mathrm{dist}(t,\Delta)} \sum_{i=n'+1}^{n} M^i \to 0 \quad \text{as} \quad n,n' \to \infty,$$

where $M = \mathrm{diam}(\Delta)/\mathrm{dist}(\Delta,t) < 1$. This proves (3.6).

So, for each $n \in \mathbb{Z}_+$ fixed we can write

$$\frac{1}{z-x} = \sum_{i=0}^{\infty} p_{\mu_n,i}(z) \int \frac{p_{\mu_n,i}(x)}{x-t} d\mu_n(x) = \sum_{i=0}^{\infty} \frac{p_{\mu_n,i}(z)}{p_{\mu_n,i}(t)} \int \frac{p^2_{\mu_n,i}(x)d\mu_n(x)}{x-t}.$$

Then

$$\varepsilon_{n,n,\infty} = \left| S_{1/(z-t),n,\mu_n} - \frac{1}{z-t} \right| = \left| \sum_{i=n+1}^{\infty} \frac{p_{\mu_n,i}(z)}{p_{\mu_n,i}(t)} \int \frac{p^2_{\mu_n,i}(x)d\mu_n(x)}{x-t} \right|.$$

Taking again into account Lemma 3.1 we see that there exists a nonnegative integer N' such that for all $n \geq N'$

$$\varepsilon_{n,n,\infty} \leq \frac{1}{\mathrm{dist}(t,\Delta)} \sum_{i=n}^{\infty} M^i \to 0 \quad \text{as} \quad n \to \infty.$$

This proves (3.5) and completes the proof of Proposition 3.1. $\square$

Recall the definition of Nikishin-Christoffel coefficients introduced in Section 2.

Proposition 3.2. *Let* $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ *and* $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *be given. Set* $k = 1$ *if* $n_1 + 1 = M = \max\{n_1 + 1, n_2 \ldots n_m\}$*, otherwise* k *is equal to the subscript of the first component of* $\mathbf{n}$ *such that* $M = n_k$*. For each* $n \in \mathbb{Z}_+$*, denote* $d\mu_{\mathbf{n}} = ds_k/Q_{\mathbf{n},2}$*. Then, for each* $j = 1, \ldots, m$*, the Nikishin-Christoffel coefficients can be written as follows*

$$\lambda_{i,j,\mathbf{n}} = \frac{||Q_{\mathbf{n}}||_{2,\mu_{\mathbf{n}}} S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}, \quad i = 1, \ldots, |\mathbf{n}|. \tag{3.8}$$

When $j = k$*, the Nikishin-Christoffel coefficients acquire the following form*

$$\lambda_{i,k,\mathbf{n}} = \frac{||Q_{\mathbf{n}}||_{2,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}, \quad i = 1, \ldots, |\mathbf{n}|. \tag{3.9}$$

Thus

$$\mathit{sign}(\lambda_{i,k,\mathbf{n}}) = \mathit{sign}(s_k), \quad i = 1, \ldots, |\mathbf{n}|. \tag{3.10}$$

In particular

$$\sum_{i=1}^{|\mathbf{n}|} |\lambda_{i,k,\mathbf{n}}| = ||s_k|| < +\infty. \tag{3.11}$$

Proof. Let us rewrite (2.5) for each $j = 1, \ldots, m$ and each $i = 1, \ldots, |\mathbf{n}|$ as

$$\lambda_{i,j,\mathbf{n}} = \int \frac{Q_{\mathbf{n}}(x) ds_j(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} = \int \frac{Q_{\mathbf{n}}(x)}{Q'_{\mathbf{n}}(x_{\mathbf{n},i})(x - x_{\mathbf{n},i})} \frac{\widehat{s}_{2,j}(x)}{\widehat{s}_{2,k}(x)} Q_{\mathbf{n},2}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)} =$$

$$\frac{||Q_{\mathbf{n}}||_{2,\mu_{\mathbf{n}}}}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})} \times$$

$$a_{\mu_{\mathbf{n}},|\mathbf{n}|} \int \frac{p_{\mu_{\mathbf{n}},|\mathbf{n}|}(x) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}{x - x_{\mathbf{n},i}} \frac{\widehat{s}_{2,j}(x)}{\widehat{s}_{2,k}(x)} Q_{\mathbf{n},2}(x) \frac{ds_k(x)}{Q_{\mathbf{n},2}(x)}.$$

Using the formula given in (3.4) it follows that

$$\lambda_{i,j,\mathbf{n}} = \frac{||Q_{\mathbf{n}}||_{2,\mu_{\mathbf{n}}} S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n}}}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|-1} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}.$$

When $j = k$, since $\widehat{s}_{2,j}/\widehat{s}_{2,k} \equiv 1$ and $\deg Q_{\mathbf{n},2} = |\mathbf{n}| - n_k$

$$\lambda_{i,k,\mathbf{n}} = \frac{||Q_{\mathbf{n}}||_{2,\mu_{\mathbf{n}}} Q_{\mathbf{n},2}(x_{\mathbf{n},i})}{a_{\mu_{\mathbf{n}},|\mathbf{n}|} Q'_{\mathbf{n}}(x_{\mathbf{n},i}) p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})}.$$

So (3.8) and (3.9) have been proved. It is well known (see [5, Theorem 5.3]) that the zeros two two consecutive elements of a family of orthogonal polynomials interlace, then $Q'_{\mathbf{n}}(x_{\mathbf{n},i})p_{\mu_{\mathbf{n}},|\mathbf{n}|-1}(x_{\mathbf{n},i})$ must be positive. Hence for each $i = 1, \ldots, |\mathbf{n}|$ the equalities (3.9) imply

$$\mathrm{sign}(\lambda_{i,k,\mathbf{n}}) = \mathrm{sign}(a_{\mu_{\mathbf{n}},|\mathbf{n}|})\mathrm{sign}(Q_{\mathbf{n},2}) =$$

$$\mathrm{sign}(s_k)\mathrm{sign}(Q_{\mathbf{n},2})\mathrm{sign}(Q_{\mathbf{n},2}) = \mathrm{sign}(s_k).$$

Combining (2.6) and (3.10) we obtain (3.11). □

4. Proof of Theorem 1.2

We proceed as in the proof of (34) in [9, Corollary 2]. Fix $\mathbf{n} \in \mathbf{\Lambda}$. Taking into account (3.11), from (2.4) we have that for each compact set $K \subset \overline{\mathbb{C}} \setminus \Delta_1$

$$\left|\left|\frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}\right|\right|_K \leq \frac{||s_k||}{\mathrm{dist}(K, \Delta_1)}.$$

Therefore, the family of functions $\{\widehat{s}_k - P_{\mathbf{n},k}/Q_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{\Lambda}}$, is uniformly bounded on each compact $K \subset \overline{\mathbb{C}} \setminus \Delta_1$ by $2||s_k||/\mathrm{dist}(K, \Delta_1)$.

Let $t_{\mathbf{n},1} < \cdots < t_{\mathbf{n},|\mathbf{n}|-n_k}$ denote the zeros of $Q_{\mathbf{n},2}$. From Lemma 2.3 we know that $\{t_{\mathbf{n},1}, \cdots, t_{\mathbf{n},|\mathbf{n}|-n_k}\} \subset \Delta_2$ and the zeros of $Q_{\mathbf{n}}$ lie in Δ_1, and

$$\left(\frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{Q_{\mathbf{n},2}}\right)(z) = \mathcal{O}\left(\frac{1}{z^{2|\mathbf{n}|+1}}\right), \qquad z \to \infty.$$

So

$$\frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{\phi_\infty^{|\mathbf{n}|+n_k+1}\prod_{i=1}^{|\mathbf{n}|-n_k}\phi_{t_{\mathbf{n},i}}} \in \mathcal{H}\left(\overline{\mathbb{C}} \setminus \Delta_1\right).$$

Take $\rho \in (0, 1)$ such that $\gamma_\rho = \{z : |\phi_\infty(z)| = \rho\}$ satisfies that $\Delta_2 \subset \mathrm{Ext}(\gamma_\rho)$, where $\mathrm{Ext}(\gamma_\rho)$ denotes the unbounded connected component of the complement of γ_ρ. We have then

$$\left|\left|\frac{\widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}}}{\phi_\infty^{|\mathbf{n}|+n_k+1}\prod_{i=1}^{|\mathbf{n}|-n_k}\phi_{t_{\mathbf{n},i}}}\right|\right|_{\gamma_\rho} \leq \frac{2|s_k|}{\mathrm{dist}(\gamma_\rho, \Delta_1)\psi^{2|\mathbf{n}|+1}(\gamma_\rho)},$$

where

$$\psi(\gamma_\rho) = \inf\{|\phi_t(z)| : z \in \gamma_\rho, t \in \Delta_2 \cup \{\infty\}\}.$$

Considered as a function of the two variables z and t, $\phi_t(z)$ is a continuous function in $\overline{\mathbb{C}}^2$. Since $\gamma_\rho \cap \Delta_2 = \emptyset$ then $\psi(\gamma_\rho) > 0$. Fix a compact $K \subset \overline{\mathbb{C}} \setminus \Delta_1$ and take ρ sufficient by close to 1 so that $K \subset \mathrm{Ext}(\gamma_\rho)$. Since the function under the norm sign is analytic in $\overline{\mathbb{C}} \setminus \Delta_1$, from the maximum principle it follows that the same bound holds for all $z \in K$. Consequently,

$$\left\| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K \leq \frac{2|s_k|\phi_\infty^{|\mathbf{n}|+n_k+1} \prod_{i=1}^{|\mathbf{n}|-n_k} \phi_{t_{\mathbf{n},i}}}{\mathrm{dist}(\gamma_\rho, \Delta_1)\psi^{2|\mathbf{n}|+1}(\gamma_\rho)} \leq \frac{2|s_k|}{\mathrm{dist}(\gamma_\rho, \Delta_1)} \left(\frac{\kappa(K)}{\psi(\gamma_\rho)} \right)^{2|\mathbf{n}|+1},$$

taking $\kappa(K)$ as in the statement of the theorem. Therefore,

$$\limsup_{|\mathbf{n}|\to\infty} \left\| \widehat{s}_k - \frac{P_{\mathbf{n},k}}{Q_{\mathbf{n}}} \right\|_K^{1/2|\mathbf{n}|} \leq \frac{\kappa(K)}{\psi(\gamma_\rho)}.$$

So, the continuity of $|\phi_t(z)|$ in $\overline{\mathbb{C}}^2$ and the fact that $\lim_{\rho\to 1} \psi(\gamma_\rho) = 1$ prove (1.2). That $\kappa(K) < 1$ is also a consequence of the continuity of $|\phi_t(z)|$ in $\overline{\mathbb{C}}^2$. $\square$

5. Proof of Theorem 1.1

We will use the following auxiliary result.

Proposition 5.1. *Let* $(s_1, \ldots, s_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $\mathbf{\Lambda} \subset \mathbb{Z}_+^m$ *be given. Assume that* $\mathrm{diam}(\Delta_k) < \mathrm{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$. *Then there exists* $N \geq 0$ *such that for each* $\mathbf{n} \in \mathbf{\Lambda}$*, where* $|\mathbf{n}| \geq N$*, every coefficient* $\lambda_{i,j,\mathbf{n}}$, $i = 1, \ldots, |\mathbf{n}|$, $j = 1, \ldots, m$ *has the same sign as its corresponding measure* s_j.

Proof. Fix an arbitrary permutation λ of $\{1, \ldots, m\}$. Define Λ_λ as the set of all $\mathbf{n} \in \Lambda$ such that there exists $\widetilde{\mathbf{s}} = (r_1, \ldots, r_m) = \mathcal{N}(\rho_1, \ldots, \rho_m)$ for which $Q_{\mathbf{n}}$ is orthogonal with respect to $(\mathbf{s}, \mathbf{n})$ and $(\widetilde{\mathbf{s}}, \widetilde{\mathbf{n}})$ (recall that $\widetilde{\mathbf{n}} = (n_{\lambda(1)}, \ldots, n_{\lambda(m)})$) in such a way that $n_{\lambda(1)} + \delta_{\lambda(1),1} \geq n_{\lambda(2)} \geq \cdots \geq n_{\lambda(m)}$. According to Lemma 2.2 we have that $\cup_\lambda \Lambda_\lambda = \Lambda$. Some of the sets Λ_λ may be empty or have a finite number of elements. Since the group of permutations of $\{1, \ldots, m\}$ is finite it is sufficient to prove that the result holds true for all λ such that Λ_λ has an infinite number of multi-indices. In the sequel we restrict our attention to such λ's and fix one of them.

Fix $\mathbf{n} \in \Lambda_\lambda$. Let us denote the measures introduced in (2.13) as

$$d\mu_{\mathbf{n},1} = \frac{d\rho_1}{Q_{\mathbf{n},2}} = \frac{ds_k}{Q_{\mathbf{n},2}} \quad \text{and} \quad d\mu_{\mathbf{n},2}(t) = \int \frac{Q_{\mathbf{n}}^2(x)}{t-x} \frac{d\rho_1(x)}{Q_{\mathbf{n},2}(x)} \frac{d\rho_2(t)}{Q_{\mathbf{n}}(t) Q_{\mathbf{n},3}(t)}. \tag{5.1}$$

We call $k = \lambda(1)$. From identities (3.8) in Proposition 3.2 it is sufficient to show that for each $j = 1, \ldots, k-1, k+1, \ldots, m$ the sequence of functions $\{S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},\mathbf{1}}}\}_{\mathbf{n}\in\Lambda_\lambda}$ converges unifromly to $\widehat{s}_{2,j}/\widehat{s}_{2,k}$ on Δ_1 because this function has constant and constant sign and no zero on Δ_1.

Denote

$$\mathcal{K}(z, x, |\mathbf{n}| - 1) = \frac{p_{\mu_{\mathbf{n},1},|\mathbf{n}|}(z)p_{\mu_{\mathbf{n},1},|\mathbf{n}|-1}(x) - p_{\mu_{\mathbf{n},1},|\mathbf{n}|}(x)p_{\mu_{\mathbf{n},1},|\mathbf{n}|-1}(z)}{z - x}.$$

Let us start by analyzing the case when $j = 1$. Taking into account the formula (3.4) and unsing the identity (2.7) in Lemma 2.1 we have that

$$\left|\frac{S_{Q_{\mathbf{n},2}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},\mathbf{1}}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{1}{\widehat{s}_{2,k}(z)}\right| =$$

$$\left|\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(\frac{Q_{\mathbf{n},2}(x)}{\widehat{s}_{2,k}(x)} - \frac{Q_{\mathbf{n},2}(z)}{\widehat{s}_{2,k}(z)}\right)d\mu_{\mathbf{n},1}(x)\right| =$$

$$\left|\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(Q_{\mathbf{n},2}(x)\ell_{2,k}(x) - Q_{\mathbf{n},2}(z)\ell_{2,k}(z)\right)d\mu_{\mathbf{n},1}(x) + \right.$$

$$\left.\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)\right)d\mu_{\mathbf{n},1}(x)\right|.$$

Since $\deg Q_{\mathbf{n},2}\ell_{2,k} \leq |\mathbf{n}| - n_k + 1 < |\mathbf{n}| - 1$ ($n_k = \max\{n_1, \ldots, n_m\}$), then

$$\left|\frac{S_{Q_{\mathbf{n},2}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},\mathbf{1}}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{1}{\widehat{s}_{2,k}(z)}\right| = |\ell_{2,k}(z) - \ell_{2,k}(z) +$$

$$\left.\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)\right)d\mu_{\mathbf{n},1}(x)\right| =$$

$$\left|\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,k}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,k}(z)\right)d\mu_{\mathbf{n},1}(x)\right|.$$

Proceeding analogously as above, for $j = 2, \ldots, k-1, k+1, \ldots, m$, and taking into account (3.4) and (2.8), we obtain

$$\left|\frac{S_{Q_{\mathbf{n},2}\widehat{s}_{2,j}/\widehat{s}_{2,k},|\mathbf{n}|-1,\mu_{\mathbf{n},\mathbf{1}}}(z)}{Q_{\mathbf{n},2}(z)} - \frac{\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)}\right| =$$

$$\left|\frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)}\int \mathcal{K}(z, x, |\mathbf{n}| - 1)\left(\frac{Q_{\mathbf{n},2}(x)\widehat{s}_{2,j}(x)}{\widehat{s}_{2,k}(x)} - \frac{Q_{\mathbf{n},2}(z)\widehat{s}_{2,j}(z)}{\widehat{s}_{2,k}(z)}\right)d\mu_{\mathbf{n},1}(x)\right| =$$

$$\left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)\right) d\mu_{\mathbf{n},1}(x) \right|.$$

Summarizing, for each $j = 1, \ldots, k-1, k+1, \ldots, m$, we need to analyze the expression

$$\left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)\right) d\mu_{\mathbf{n},1}(x) \right|.$$

Using Fubini's Theorem we obtain the following chain of equalities

$$\begin{aligned}
&\left| \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(Q_{\mathbf{n},2}(x)\widehat{\tau}_{2,j}(x) - Q_{\mathbf{n},2}(z)\widehat{\tau}_{2,j}(z)\right) d\mu_{\mathbf{n},1}(x) \right| = \\
&\left| \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{Q_{\mathbf{n},2}(x)}{x-t} - \frac{Q_{\mathbf{n},2}(z)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) \right| = \\
&\left| \int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{Q_{\mathbf{n},2}(x) - Q_{\mathbf{n},2}(t)}{x-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) - \right.\\
&\int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{Q_{\mathbf{n},2}(z) - Q_{\mathbf{n},2}(t)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) + \\
&\int \frac{a_{\mu_{\mathbf{n},1},|\mathbf{n}|}}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{Q_{\mathbf{n},2}(t)}{x-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) - \\
&\left. \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{Q_{\mathbf{n},2}(t)}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) \right| = \\
&\left| \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} a_{\mu_{\mathbf{n},1},|\mathbf{n}|} \int \mathcal{K}(z,x,|\mathbf{n}|-1) \left(\frac{1}{x-t} - \frac{1}{z-t} \right) d\mu_{\mathbf{n},1}(x) d\tau^k_{2,j}(t) \right| = \\
&\left| \int \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \left(S_{1/(z-t),|\mathbf{n}|-1,\mu_{\mathbf{n},1}} - \frac{1}{z-t} \right) d\tau^k_{2,j}(t) \right| \leq \\
&\left\| \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \right\|_{S(\sigma_2)} \left\| S_{1/(z-t),|\mathbf{n}|-1,\mu_{\mathbf{n},1}} - \frac{1}{z-t} \right\|_{\Delta_1} \left\| \tau^k_{2,j} \right\|.
\end{aligned}$$

Combining the requirement $\operatorname{diam}(\Delta_k) < \operatorname{dist}(\Delta_1, \Delta_2)$, $k = 1, 2$, Lemma 3.1 and Proposition 3.1 we obtain that

$$\left\| \frac{Q_{\mathbf{n},2}(t)}{Q_{\mathbf{n},2}(z)} \right\|_{S(\sigma_2)} \to 0 \quad \text{and} \quad \left\| S_{1/(z-t),|\mathbf{n}|-1,\mu_{\mathbf{n},1}} - \frac{1}{z-t} \right\|_{\Delta_1} \to 0.$$

So this completes the proof. $\square$

Now we are ready to prove Theorem 1.1. As in Section 4, we take $\rho \in (0,1)$ and $\gamma_\rho = \{z : |\phi_\infty(z)| = \rho\}$. For each $j = 1, \ldots, k-1, k+1, \ldots, m$ we have that

$$||\widehat{s}_j||_{\gamma_\rho} = \frac{|s_j|}{\mathrm{dist}(\gamma_\rho, \Delta_1)} \quad \text{and} \quad \left|\left|\frac{P_j}{Q_{\mathbf{n}}}\right|\right|_{\gamma_\rho} = \left|\left|\sum_{i=1}^{|\mathbf{n}|} \frac{\lambda_{i,j,\mathbf{n}}}{z - x_{\mathbf{n},i}}\right|\right|_{\gamma_\rho} \leq \frac{|s_j|}{\mathrm{dist}(\gamma_\rho, \Delta_1)}$$

The second inequality can be deduced easily from Proposition 5.1. Combining the above inequalities we have that

$$\left|\left|\frac{\widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}}}{\phi_\infty^{|\mathbf{n}|+n_j+1}}\right|\right|_{\gamma_\rho} \leq \frac{2|s_j|}{\mathrm{dist}(\gamma_\rho, \Delta_1)\rho^{|\mathbf{n}|+n_j+1}}.$$

Let us fix a compact $K \subset \bar{\mathbb{C}} \setminus \Delta_1$ and take ρ sufficient close to 1. From the maximum principle it follows that the same bound holds for all $z \in K$. Consequently,

$$\left|\left|\widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}}\right|\right|_K \leq \frac{2|s_j|\,||\phi_\infty||_K^{|\mathbf{n}|+n_j+1}}{\mathrm{dist}(\gamma_\rho, \Delta_1)\rho^{|\mathbf{n}|+n_j+1}}.$$

Therefore,

$$\limsup_{|\mathbf{n}|\to\infty} \left|\left|\widehat{s}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}}\right|\right|_K^{1/(|\mathbf{n}|+n_j)} \leq \frac{||\phi_\infty||}{\rho},$$

and the result readily follows making $\rho \to 1$.

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